PHILOSOPHICAL TRANSACTIONS.

Monday, *April* 13. 1668

The Contents.

The Squaring of the Hyperbola, *by an infinite series of Rational Numbers, together with its Demonstration, by that Eminent Mathematician, the Right Honourable the Lord Viscount* Brouncker.

WHat the Acute Dr. *John Wallis* had intimated, some years since, in the Dedication of his Answer to M. *Meibomius de proportionibus, vid.* That the World one day would learn from the Noble Lord *Brounker*, the *Quadrature* of the *Hyperbole*; the Ingenious Reader may see performed in the subjoyned operation, which its Excellent Author was now pleased to communicate, as followeth in his own words;

My Method for Squaring the Hyperbola *is this:*

LEt AB be one *Asymptote* of the Hyperbola EdC; and let AE and BC be parallel to th'other: Let also AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See *Fig.* 1. And note, that the Letter x every where stands for Multiplication.

Supposing the Reader knows, that EA. αζ. KH. βη. dθ. γκ. δλ. εμ. CB.&c. are in an *Harmonic* series, or a *series reciproca primanorum seu arithmetice proportionalium* (otherwise he is referr'd for satisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. *Arithm. Infinitor. Wallisij*:)

I say $ABCdEA = \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} + \frac{1}{7 \times 8} + \frac{1}{9 \times 10}$ &c.

$EdCDE = \frac{1}{2 \times 3} + \frac{1}{4 \times 5} + \frac{1}{6 \times 7} + \frac{1}{8 \times 9} + \frac{1}{10 \times 11}$ &c. *in infinitum.*

$EdCyE = \frac{1}{2 \times 3 \times 4} + \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} + \frac{1}{8 \times 9 \times 10}$ &c.

For (in Fig. 2, & 3) the Parallelog.		*And (in Fig. 4.) the Triangl.*	*Note.*
$CA = \frac{1}{1 \times 2}$		$EdC = \frac{1}{2 \times 3 \times 4} = \frac{\square dD - \square dF}{2}$	
$dD = \frac{1}{2 \times 3}$	$dF = \frac{1}{3 \times 4}$	$Ebd = \frac{1}{4 \times 5 \times 6} = \frac{\square br - \square bn}{2}$	$\frac{1}{2}CA = dD + dF$
$br = \frac{1}{4 \times 5}$	$bn = \frac{1}{5 \times 6}$	$dfC = \frac{1}{6 \times 7 \times 8} = \frac{\square fG - \square fk}{2}$	$\frac{1}{2}dD = br + bn$
$fG = \frac{1}{6 \times 7}$	$fk = \frac{1}{7 \times 8}$	$Eab = \frac{1}{8 \times 9 \times 10} = \frac{\square aq - \square ap}{2}$	$\frac{1}{2}dF = fG + fk$
$aq = \frac{1}{8 \times 9}$	$ap = \frac{1}{9 \times 10}$	$bcd = \frac{1}{10 \times 11 \times 12} = \frac{\square cs - \square cm}{2}$	$\frac{1}{2}br = aq + ap$
$cs = \frac{1}{10 \times 11}$	$cm = \frac{1}{11 \times 12}$	$def = \frac{1}{12 \times 13 \times 14} = \frac{\square et - \square el}{2}$	$\frac{1}{2}bn = cs + cm$
$et = \frac{1}{12 \times 13}$	$el = \frac{1}{13 \times 14}$	$fgC = \frac{1}{14 \times 15 \times 16} = \frac{\square gu - \square gh}{2}$	$\frac{1}{2}fG = et + el$
$gu = \frac{1}{14 \times 15}$	$gh = \frac{1}{15 \times 16}$	&c.	$\frac{1}{2}fk = gu + gh$
&c.	&c.		&c.

And

And that therefore in the firſt ſeries half the firſt term is greater than the ſum of the two next, and half this ſum of the ſecond and third greater than the ſum of the four next, and half the ſum of thoſe four greater than the ſum of the next eight, *&c. in infinitum.* For $\frac{1}{2}$ *dD* = *br* + *bn*; but *bn* > *fG*, therefore $\frac{1}{2}$ *dD* > *br* + *fG*, *&c.* And in the *ſecond* ſeries half the firſt term is leſs then the ſum of the two next, and half this ſum leſs then the ſum of the four next, *&c. in infinitum.*

That the firſt *ſeries* are the *even* terms, *viz.* the 2^d^, 4^th^, 6^th^, 8^th^, 10^th^, *&c.* and the *ſecond*, the *odd*, *viz.* the 1^ſt^, 3^d^, 5^th^, 7^th^, 9^th^, *&c.* of the following ſeries, *viz.* $\frac{1}{1\times2}\cdot\frac{1}{2\times3}\cdot\frac{1}{3\times4}\cdot\frac{1}{4\times5}\cdot\frac{1}{5\times6}\cdot\frac{1}{6\times7}$. *&c. in infinitum* = 1. Whereof *a* being put for the number of terms taken at pleaſure, $\frac{1}{a^2+a}$ is the laſt, $\frac{a}{a+1}$ is the ſum of all thoſe terms from the beginning, and $\frac{1}{a+1}$ the ſum of the reſt to the end.

That $\frac{1}{4}$ of the firſt terme in the *third* ſeries is leſs than the ſum of the two next, and a quarter of this ſum, leſs than the ſum of the four next, and one fourth of this laſt ſum leſs than the next eight, I thus demonſtrate.

Let a = the 3^d^ or laſt number of any term of the firſt Column, *viz.* of Diviſors,

$$\frac{1}{a \times a-1 \times a-2} = \frac{1}{a^3-3a^2+2a} = \frac{16a^3-48a^2+56a-24}{16a^6-96a^5+232a^4-288a^3+184a^2-48a} = A$$

$$\left.\begin{array}{l}\frac{1}{2a \times 2a-1 \times 2a-2} = \frac{1}{8a^3-12a^2+4a} \\ \frac{1}{2a-2 \times 2a-3 \times 2a-4} = \frac{1}{8a^3-36a^2+52a-24}\end{array}\right\} = \frac{16a^3-48a^2+56a-24}{64a^6-384a^5+880a^4-960a^3+496a^2-96a} = B$$

$$\frac{64a^6-384a^5+928a^4-1152a^3+736a^2-192a}{64a^6-384a^5+880a^4-960a^3+496a^2-96a} \times \tfrac{1}{4} A < B.$$

And $48a^4-192a^3+240a^2-96a$ = *Exceſs of the Numerator above Denomin.*

$$\left.\begin{array}{lcl}\text{But --- The affirm.} & > & \text{the Negat.} \\ \text{That is, } 48a^4+240a^2 & > & 192a^3+96a \\ \text{Becauſe } a^4+5a^2 & > & 4a^3+2a \\ a^3+5a & > & 4a^2+2\end{array}\right\} \text{if } a > 2.$$

Therefore B > $\frac{1}{4}$ A.

Therefore $\frac{1}{4}$ of any number of A: or Terms, is leſs than their ſo many reſpective B. that is, than twice ſo many of the next Terms. *Quod, &c.*

By

By any one of which three Series, it is not hard to calculate, as near as you please, these and the like *Hyperbolic* spaces, whatever be the *Rational* Proportion of *A E* to *B C*. As for Example, when *A E* is to *B C*, as 5 to 4. (whereof the Calculation follows after that where the Proportion is, as 2 to 1. and both by the third Series.)

First then when (in Fig. 1.) AE. BC :: 2. 1.

2x3x4)	1.	(0.0416666666 —	0.0416666666
4x5x6)	1.	(0.0083333333 —	0.0113095237
6x7x8)	1.	(0.0029761904 —	
8x9x10)	1.	(0.0013888888 —	0.0029019589
10x11x12)	1.	(0.0007575757 —	
12x13x14)	1.	(0.0004578754 —	
14x15x16)	1.	(0.0002976190 —	
16x17x18)	1.	(0.0002042484 —	0.0007306482
18x19x20)	1.	(0.0001461988 —	
20x21x22)	1.	(0.0001082251 —	
22x23x24)	1.	(0.0000823452 —	
24x25x26)	1.	(0.0000641026 —	
26x27x28)	1.	(0.0000508751 —	
28x29x30)	1.	(0.0000410509 —	
30x31x32)	1.	(0.0000336021 —	
32x33x34)	1.	(0.0000278520 —	0.0001829939
34x35x36)	1.	(0.0000233426 —	
36x37x38)	1.	(0.0000197566 —	
38x39x40)	1.	(0.0000168691 —	
40x41x42)	1.	(0.0000145180 —	
42x43x44)	1.	(0.0000025843 —	
44x45x46)	1.	(0.0000109793 —	
46x47x48)	1.	(0.0000096361 —	
48x49x50)	1.	(0.0000085034 —	
50x51x52)	1.	(0.0000075415 —	
52x53x54)	1.	(0.0000067193 —	
54x55x56)	1.	(0.0000060125 —	
56x57x58)	1.	(0.0000054014 —	
58x59x60)	1.	(0.0000048704 —	
60x61x62)	1.	(0.0000044068 —	
62x63x64)	1.	(0.0000040002 —	

0.0416666666
0.0113095237
0.0029019589
0.0007306482
3)0.0001829939(0.0000609980
0.05679179
+ 0.00006100
0.05685279 < E d C y

But
0.0007306482
0.0001829939
0.0000458315

Therefore 0.05679179
+ 0.00004583
+ 0.00001528
0.05685290 > E d C y.

For, it has been demonstrated that ⅓ of any terme in the last Column is less than the terme next after it; and therefore that ⅓ of the last terme, at which you stop

ſtop, is leſs than the remaining terms, and that the total of theſe is leſs than $\frac{4}{3}$ of a third proportional to the two laſt.

And therefore A B C y E being = 0.75 ———————— 0.75
and E d C y > 0.05685279 ——— and < 0.05685290

And A BCdE is < 0.69314720 ——— and > 0.69314709

But when AE . BC:: 5. 4. or as EA. to KH. then will the ſpace A B C E. or now, the ſpace AHKE (AH = $\frac{1}{4}$AB.) be found as follows.

8 x 9x10) 1 (0.0013888888
16x17x18) 1 (0.0002042484 } 0.0003504472
18x19x20) 1 (0.0001461988 }
32x33x34) 1 (0.0000278520 }
34x35x36) 1 (0.0000233426 } 0.0000878204
36x37x38) 1 (0.0000197566 }
38x39x40) 1 (0.0000168691 }

0.00 3888888
0.0003504472
3)0.0000878204(0.0000292735
0.0018271564
+ 0.0000292735
0.0018564299 < E a b

But 0.0003504472 }
0.0000878204 } ∴
0.0000220073 }

Therefore 0.0018271564
+ 0.0000220074
+ 0.0000073358
0.0018564996 > E a b

Therefore E M b. (Fig 4.)
being = 0.025 ———————— 0.025
E a b > 0.0018564299 —— & < 0.0018564996

EMba (Fig.4.) or EKM (Fig. 1.) > 0.02685643 ——— < 0.02685650
A H K M < 0.22314356 ——— > 0.22314349

Therefore 3 A B C d E = 2.07944154
and A H K E. = 0.2231435 ———
ABCdE (when AE.BC:: 10.1.) = 2. 025850 ———

Therefore the Logar. of 10 is to the Log. of 2, as 2.302585 to 0.693147

Fig. I.

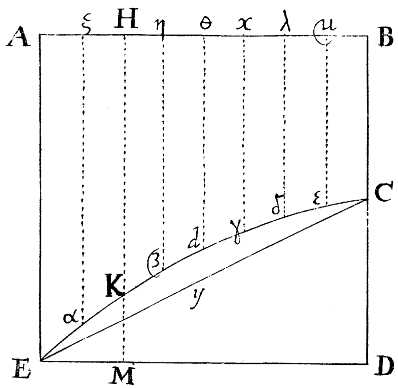

Fig. II.

Fig. III.

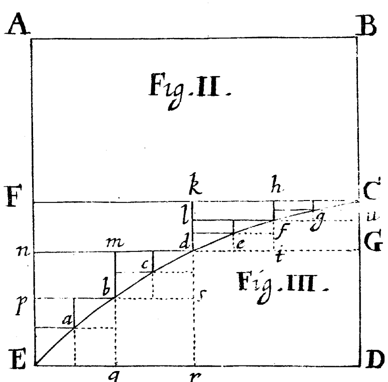

Fig. V.

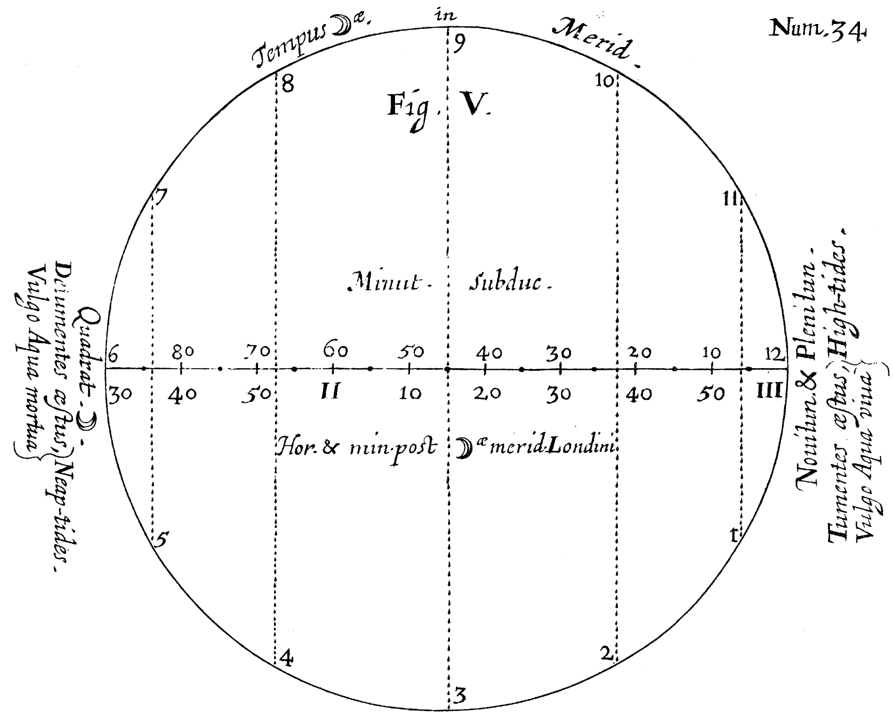

Fig. IIII.